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Monte Carlo simulations of the Ising spin glass on lattices with finite connectivity

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Abstract. We simulated the Ising spin-glass model on a random lattice with a finite (average) coordination number and also on the Bethe lattice with various different boundary conditions. In particular, we calculated the overlap function $P(q)$ for two independent samples. For the random lattice, the results are consistent with a spin-glass transition above which $P(q)$ converges to a Dirac δ function for large N (number of sites) and below which $P(q)$ has in addition a long tail similar to previous results obtained for the infinite-range model. For the Bethe lattice, we obtain results in the interior by discarding the two outer shells of the Cayley tree when calculating the thermal averages. For fixed (uncorrelated) boundary conditions, $P(q)$ seems to converge to a δ function even below the spin-glass transition whereas on a 'closed' lattice (correlated boundary conditions) $P(q)$ has a long tail similar to its behaviour in the random-lattice case.

1. Introduction

The theory of randomly frustrated systems such as the spin glass has been fully understood only at the mean-field level in which the interaction range is infinite. The Parisi solution [1] leads to the many-pure-state picture in the spin-glass phase. However, there has been much controversy over the existence of many pure states in the short-range case [2, 3]. The long-range Sherrington–Kirkpatrick (SK) model is 'universal' in the sense that it just depends on the first two cumulants of the bond distribution [4]. But the short-range model depends on the type of bond distribution, i.e. it depends on all the cumulants of the bond distribution. To tackle the short-range model, one might think of using a loop expansion. The first step is the zero-loop or tree approximation which is known to be unstable without replica symmetry breaking (RSB) for a Gaussian bond distribution. Even at this level of approximation one has to introduce a sequence of order parameters q_{α}, $q_{\alpha\beta}$, $q_{\alpha\beta\gamma}, \ldots$, etc for non-Gaussian bond distribution (for Gaussian, q_{α} and $q_{\alpha\beta}$ are sufficient) and there is no known scheme of introducing RSB for these order parameters.

In this paper, we first consider the Viana–Bray [5] model of the random lattice. This lattice consists of N sites and a bond connecting two sites is placed with a probability $p = c/N$ with $c = \mathrm{O}(1)$ with respect to N. Thus the lattice has a finite coordination number and is locally isomorphic to a tree since the probability of finding a loop of small size is $\mathrm{O}(1/N)$, and hence the tree approximation is exact [6] up to $\mathrm{O}(1/N)$. Viana and Bray, using a perturbative expansion, showed that the replica symmetric solution is unstable just below the spin-glass transition. Mézard and Parisi [6] and Kanter and Sompolinksy [7] found a replica symmetric solution to the model at $T=0$ using the effective field method. This solution was subsequently shown [8] to be unstable to RSB (and also to the inclusion of a continuous part), but an explicit solution which breaks replica symmetry has not yet been found due to the complexity of the problem (infinite number of order parameters $q_{\alpha\beta}$, $q_{\alpha\beta\gamma}, \ldots$, etc). We simulate and study the properties of the spin glass on this lattice. The probability distribution of the overlap function $P(q)$ is computed. The results will provide some valuable information on the nature of the RSB and the structure of the phase space.

The spin-glass model on the Bethe lattice has been investigated extensively in recent years [9-13]. The Bethe lattice has a finite coordination number and thus may be a better starting point for the understanding of the short-ranged spin glass than the infinite-range model. Surprisingly though, this model was found to possess many similar properties to the infinite-range model. The Bethe lattice has some subtleties in the sense that the properties of the solutions may depend on the choice of boundary conditions. Distinctions have been made in the literature between 'uncorrelated' and 'correlated' boundary conditions. The model with correlated boundary conditions has been shown recently to undergo RSB in the spin-glass phase close to and below the spin-glass transitions [13]. This suggests the existence of many coexisting thermodynamic states as in the SK model. On the other hand, for uncorrelated boundary conditions, a solution similar to the original SK model has been obtained [12] (no RSB). It is very probable, and is suggested by the results of this work, that the Bethe lattice with correlated boundary conditions is very similar to the random graph with fixed connectivity which can be viewed as a 'closed' Bethe lattice since, as mentioned before, small loops are rare ($O(1/N)$). In this context we mention that it is easy to generate a random graph with fixed connectivity [14] although in this paper we will consider the closely related random graph with average finite connectivity *à la* Viana and Bray. In this paper, we will use Monte Carlo simulations to investigate the effects of different boundary conditions on the spin-glass solution.

In addition to the theoretical interest in the short-range spin-glass problem, due to the close relationship between spin-glass systems and complex optimisation problems [15, 16], this type of model is of much practical importance for the problem of bipartitioning of a graph with finite connectivity [16]. A better understanding of the nature of the phase space will give valuable information needed for computer optimisation problems.

2. Theory

2.1. The random-lattice spin glass

We consider the Ising spin-glass system described by the Hamiltonian

$$H=-\sum_{(ij)} J_{ij}S_iS_j - h_{\text{ext}}\sum_i S_i \tag{2.1}$$

where $S_i=\pm 1$, J_{ij} is the random interaction between a pair of spins (i,j) and h_{ext} is the constant external field.

For the random graph model, J_{ij} is long ranged with probability distribution given by

$$\mathscr{P}(J_{ij})=(1-c/N)\delta(J_{ij})+(c/N)P(J_{ij}) \tag{2.2}$$

where N is the total number of spins and c is the average number of neighbours which is of order 1; i.e. average finite connectivity. $P(J_{ij})$ is the probability distribution of the surviving bonds. The SK model can be recovered with $c \to N$ and proper scaling of the J_{ij}. Mean-field theory of this model is believed to be exact because of the local tree-like structure of the random lattice. (Small loops are rare and frustration comes mainly from loops of size $O(N)$.)

Viana and Bray [5] showed that, using the replica method, the free energy per spin is given by

$$\begin{aligned} \beta f\{q\} &= \lim_{n\to\infty} \frac{1}{n}\left(\frac{a_1}{2}\sum_{\alpha} q_\alpha^2 + \frac{a_2}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 + \frac{a_3}{2}\sum_{\alpha<\beta<\gamma} q_{\alpha\beta\gamma}^2 + \ldots\right) - \frac{1}{n}\ln \operatorname{Tr}\exp(\chi) \\ \chi &= \beta h_{\text{ext}}\sum_{\alpha} S^\alpha + a_1\sum_{\alpha} q_\alpha S^\alpha + a_2 \sum_{\alpha<\beta} q_{\alpha\beta}S^{\alpha\beta} + \ldots \end{aligned} \tag{2.3}$$

where $a_k = c\int \mathrm{d}J\, P(J)\tanh^k(\beta J)$ and $\alpha, \beta, \gamma, \ldots$ are replica indices.

The q_α, $q_{\alpha\beta}$, $q_{\alpha\beta\gamma}$, etc take values that minimise f. The set of infinite-order parameters q makes this type of short-ranged spin-glass problem difficult; however, near the transition temperatures one can neglect all other parameters but those with a few indices. The model then shows a similar structure to that of the SK model which has only q_α and $q_{\alpha\beta}$, with the correspondence

$$a_2 = \beta_{\text{SK}}^2 \qquad \text{near } a_2 = 1. \tag{2.4}$$

The Almeida-Thouless (AT) instability line, below which the replica symmetric solution is unstable, is given near transition by [5]

$$\left(\frac{\beta h_{\text{ext}}}{1-a_1}\right)^2 = \left(\frac{1}{6} + \frac{a_4}{2(1-a_4)}\right)(a_2-1)^3. \tag{2.5}$$

2.2. *The Bethe lattice*

The Bethe lattice is a tree-like structure in which every site is connected to a finite fixed number of neighbours. The lattice is constructed as follows. One starts with a central site 0; then c sites are connected to 0 to form the first level. A further level is built by joining $c-1$ new sites to every site on the outermost level. Continuing this process, one constructs a graph with n levels. This graph is called a Cayley tree.

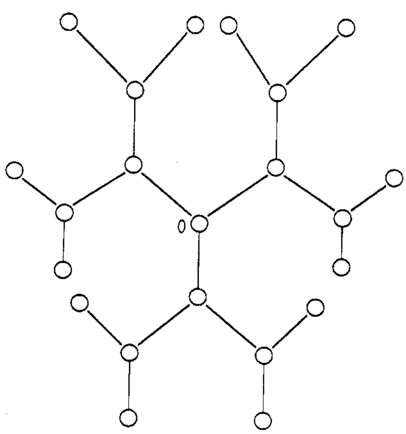

Figure 1. Cayley tree with three levels and $c = 3$.

Figure 1 shows a Cayley tree with three levels. This graph has no closed loops and if one ignores the boundary sites, this graph is a regular graph on which every site has the same coordination number c. The Bethe lattice is the interior portion of a Cayley tree with $n \to \infty$, i.e. only sites deep within the graph and very far away from the boundary are included.

As mentioned in the introduction, the model has been studied analytically [9-13] and the existence of the spin-glass phase was shown. The tree-like structure allows one to write down recursion relations expressing properties of the nth level in terms of those of the $(n+1)$th level. Frustration is introduced by the effect of the boundary conditions. Ideally, the bulk properties of the Bethe lattice can be explored by considering a Cayley tree with a very large number of levels, say n_1, and one only looks at the thermal ensemble of the sites in the interior portion of the first n_2 levels. Then the limit $n_1 \to \infty$ is taken before $n_2 \to \infty$. However, in practical simulations, this is not easy to achieve because the number of sites on the nth level is

$$c[c-1]^{n-1} \tag{2.6}$$

which grows exponentially as n. Hence it is not economical and not affordable to discard a large number of the outermost layers from the boundary.

In a short-ranged spin model on a regular lattice, boundary conditions can be used to project one or more states out of several stable thermodynamic states which coexist in the ordered state. In the case of a spin-glass model on a Bethe lattice, boundary conditions are also necessary to introduce frustration into the model. It is possible that some boundary conditions will accommodate only one thermodynamic state as claimed by Carlson *et al* [12] for uncorrelated boundary conditions, whereas other boundary conditions will accommodate many thermodynamic states, as given by the Mottishaw solution [13].

To check this possibility we considered two types of boundary conditions on a finite Cayley tree: (i) fixed boundary conditions which belong to the 'uncorrelated' type and (ii) closed boundary conditions (we actually closed the tree) which belong to the class of correlated boundary conditions. In order to obtain results which are relevent for the Bethe lattice (i.e. the interior of a very large Cayley tree) we discarded, when performing thermal averages, the boundary layer and one extra layer for both boundary conditions. The number of sites thus taken in calculating the thermal averages are about 25% (for $c=3$) of the total number of sites of the Cayley tree. The more layers discarded the better the approximation to the Bethe lattice but the fewer the number of spins remaining for the thermal averages.

In principle it is possible to consider the finite Cayley tree without discarding the outside layers provided one imposes temperature-dependent boundary conditions that emulate the effect of the infinite tree on the internal finite part. But this is very difficult to achieve in practice since one has to know *a priori* the analytic solution for the effective fields in the Bethe lattice which is not available for the case of correlated boundary conditions.

We should emphasise here that even for the infinite Bethe lattice one has to specify the boundary conditions. This is because the properties near the centre may depend on boundary conditions very far away. Thus when taking the thermodynamic limit one has still to specify the boundary conditions on the circumference of the tree. This point becomes clear in the analytic solution of the recursion relations. One looks for fixed-point solutions of these recursion relations, thus imposing shell independence, which corresponds to the behaviour in the interior of an infinite tree. Nevertheless, some fixed-point solutions may be unstable thermodynamically, and it is the boundary conditions on the infinite tree that determine which solutions are acceptable, and which are incompatible with the boundary conditions.

2.3. *The overlap function $P(q)$*

The question of interest is whether the many-pure-state structure of the SK model remains true for the present finite-connectivity models. It was shown that near transition the random lattice is qualitatively the same as the SK model and hence the many-valley phase space structure should be true at least just below transition.

Following the treatment of the SK model, one can define the overlap between different valleys l and l' by

$$q^{ll'} = \frac{1}{N} \sum_{i=1}^{N} m_i^l m_i^{l'} \tag{2.7}$$

where m_i^l is the magnetisation of site i in the state l. The probability distribution for an overlap q is given by

$$P(q) = \left\langle \sum_{ll'} P_l P_{l'} \delta(q - q^{ll'}) \right\rangle_J \tag{2.8}$$

where P_l is the weight in state l. In the SK model $P(q)$ is related to the Parisi order parameter function $q(x)$ via [17]

$$P(q) = \mathrm{d}x(q)/\mathrm{d}q \tag{2.9}$$

where $x(q)$ is the inverse function of $q(x)$.

In the present model, in which there are infinitely many order parameters q_α, $q_{\alpha\beta}$, $q_{\alpha\beta\gamma}\ldots$, one does not know how to construct a similar quantity $q(x)$ as in the SK model. However, the overlap probability distribution $P(q)$ is still well defined in terms of the many-states phase space structure. For a finite number of states, $P(q)$ is characterised by a sum of a finite number of δ functions. On the other hand, a continuous part in $P(q)$ usually suggests the existence of many states in phase space. (There are pathological cases [2] for which this correspondence may not hold.)

In actual simulation with finite-size systems, one can define

$$P_N(q) = \left\langle \left\langle \delta\left(q - \frac{1}{N} \sum_i S_i^{(1)} S_i^{(2)} \right) \right\rangle_{\mathrm{th}} \right\rangle_J \tag{2.10}$$

where $S_i^{(1)}$ and $S_i^{(2)}$ are two identical but independent spin systems which have the same set of couplings J_{ij}. The subscript in $\langle\ \rangle_{\mathrm{th}}$ denotes the thermal average (which amounts to a time average in a Monte Carlo simulation), while $\langle\ \rangle_J$ denotes the quench average of the J_{ij} configurations. It can be shown that [18]

$$\lim_{N\to\infty} P_N(q) = P(q). \tag{2.11}$$

These two independent sets of spins are simulated for a time t_0 until thermal equilibrium is reached. One has to make sure that t_0 is large enough for each size. Then the thermal averages, which are time averages for $t > t_0$ in simulation, are taken. The above procedure is repeated for many other sets of realisations of the J_{ij}. Finally the quench averages of the J are performed. In the SK model, $P(q)$ is shown [19] to be non-self-averaging; thus the average in J_{ij} is essential and must be performed explicitly.

3. Results

3.1. Random graph

We consider the case of the $J = \pm 1$ spin glass, i.e.

$$P(J_{ij}) = \tfrac{1}{2}(\delta(J_{ij}+1) + \delta(J_{ij}-1)). \tag{3.1}$$

The spin-glass transition temperature is given by [5]

$$T_c = (\tanh^{-1}(1/\sqrt{c}))^{-1} \qquad \text{for} \qquad h_{\text{ext}} = 0. \tag{3.2}$$

Only the case of $c = 3$ is considered for the results of the simulations. The sizes of the systems range from $N = 16$ to $N = 100$. The results for large N were obtained from the Cray/XMP48.

To avoid the possibility of being trapped in the local minima in phase space at low temperatures, we made use of the simulated annealing technique in the simulations. The systems are initially prepared and simulated for equilibration at some high temperature (above T_c) and thermal samples are taken. Then the temperature is lowered and the above procedure is repeated. The temperature of the systems is then gradually reduced to the desired low temperature. The cooling rate should not be too rapid to avoid being trapped in the local minima. The equilibration time should be longer near the freezing temperature and more thermal samples should be taken at low temperatures. These in turn depend on the size of the system. The typical equilibration times range from 3000 to 12 000 steps and the number of thermal samples taken ranges from 300 to 1000 depending on the size of the system. Then the whole process above is repeated for a different J_{ij} configuration. The typical number of J_{ij} configurations ranges from 50 to 200 and the quenched average is then taken.

3.1.1. Critical temperature and susceptibility. In the spin-glass context, a continuous transition occurs when the susceptibility defined as

$$\chi_{\text{SG}} = (1/N) \sum_{ij} \langle \langle S_i S_j \rangle_{\text{th}}^2 \rangle_J \tag{3.3}$$

diverges. It is easy to show that

$$\chi_{\text{SG}} = N \int q^2 P_N(q)\, \mathrm{d}q. \tag{3.4}$$

Figure 2 shows a plot of χ against N for various temperatures. For $T > T_c$, χ approaches a constant as N increases, while for $T < T_c$, χ seems to diverge as N increases. From (3.2), $T_c = 1.52$ for $c = 3$ which is consistent with the value suggested from the plot.

3.1.2. Lack of self-averageness. The distribution $P(q)$ is not self-averaging [19] in the SK model, and it is believed that this non-self-averageness is due to the existence of many degnerate states which contribute to the Gibbs average. A quantity that measures the lack of self-averageness is

$$\Delta q = \left[\left\langle \left(\frac{1}{N} \sum_i \langle S_i \rangle_{\text{th}}^2 \right)^2 \right\rangle_J - \left\langle \frac{1}{N} \sum_i \langle S_i \rangle_{\text{th}}^2 \right\rangle_J^2 \right]^{1/2} \tag{3.5}$$

and self-averageness is characterised by

$$\lim_{N\to\infty} \Delta q \to 0. \tag{3.6}$$

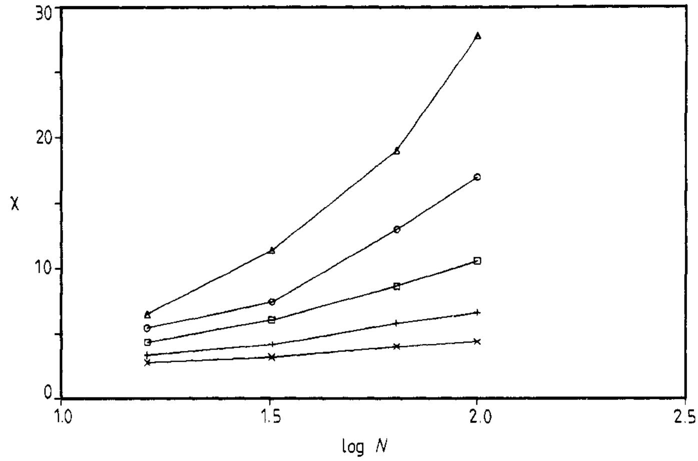

Figure 2. Plot of χ against log N at $T = 1.6$ (crosses), 1.4 (plus signs), 1.2 (squares), 1.0 (circles) and 0.8 (triangles).

Figure 3 is a plot of Δq against N for various temperatures and values of h_{ext}. The points above the AT line do show the trend that $\Delta q \to 0$ as $N \to \infty$ while those below suggest $\Delta q \to$ constant as $N \to \infty$.

3.1.3. Near T_c. From the argument in § 2, it is suggested that for $a_2 \gg 1$, the present model should have the same RSB as the SK model with the correspondence given by (2.4). One would like to see this in actual simulations. We choose $a_2 = 1.096$ which corresponds to $\beta_{\text{SK}} = 1.047$. We simulated the SK model with Gaussian J_{ij} at this temperature and compared it with the random lattice with $a_2 = 1.096$. Figure 4 shows the $P(q)$ for the two cases; they have almost identical $P(q)$. Since $P(q)$ is related to $q(x)$ via (2.9), this confirms that near and below the spin-glass transition, the random lattice shows the same fashion of RSB effect as the SK model.

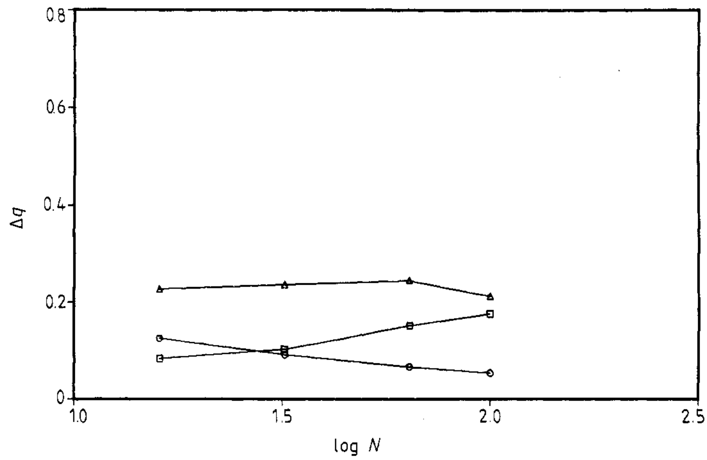

Figure 3. Plot of Δq against log N. Above the AT line, $h_{\text{ext}} = 1.5$ and $T = 1.4245$ (circles); below the AT line, $h_{\text{ext}} = 0$, $T = 0.8$ (triangles) and $T = 1.0$ (squares).

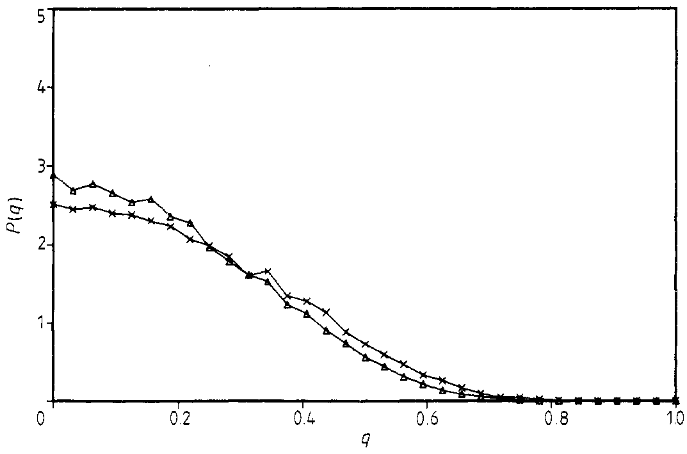

Figure 4. $P(q)$ for the SK model at $\beta = 1.047$ (triangles) and for the random lattice with $c = 3$ at $a_2 = 1.096$ (crosses). $N = 64$ and $h_{ext} = 0$ in both cases.

3.1.4. Above the AT *line.* We choose $a_2 = 1.1$ (i.e. $T = 1.4245$) which is close to and below the transition. From (2.5), the external field on the AT line at this temperature is about 2.2×10^{-2}, so we choose $h_{ext} = 1.5$ which is well above the AT line. Figure 5 shows the distribution function of the overlap for various sizes. $P_N(q)$ peaks at $q \sim 0.5$ and gets sharper as N increases. From (2.8), if there is only one single pure state, then $P(q)$ is a δ function. However in a finite system of size N, $P_N(q)$ should be a Gaussian function of width proportional to $N^{-1/2}$. Figure 6 is a plot of the standard deviation of $P_N(q)$ against $N^{-1/2}$; it shows a straight line passing through the origin and this confirms that $P_N(q)$ is a δ function as $N \to \infty$.

3.1.5. Below the AT *line.* We take $h_{ext} = 0$ and $T = 0.8$ which is well below the AT line. For $h_{ext} = 0$, time-reversal symmetry is preserved and $P(q)$ is an even function of q. Thus only the portion with $q > 0$ will be shown for the $P(q)$ curves. Figure 7 displays

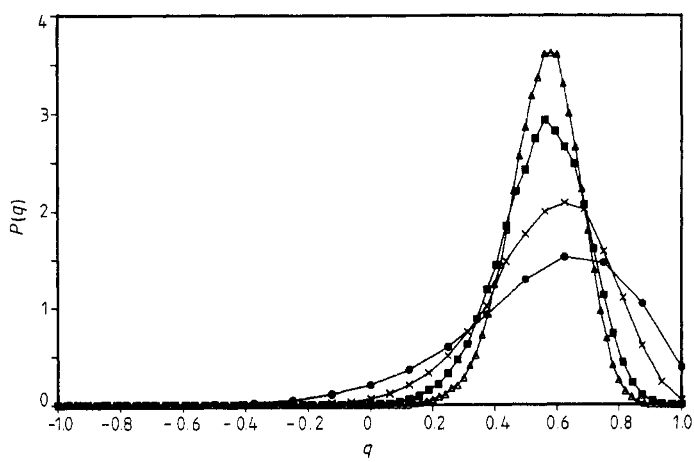

Figure 5. $P(q)$ above the AT line with $h_{ext} = 1.5$ and $T = 1.4245$ for $N = 16$ (circles), 32 (crosses), 64 (squares) and 100 (triangles).

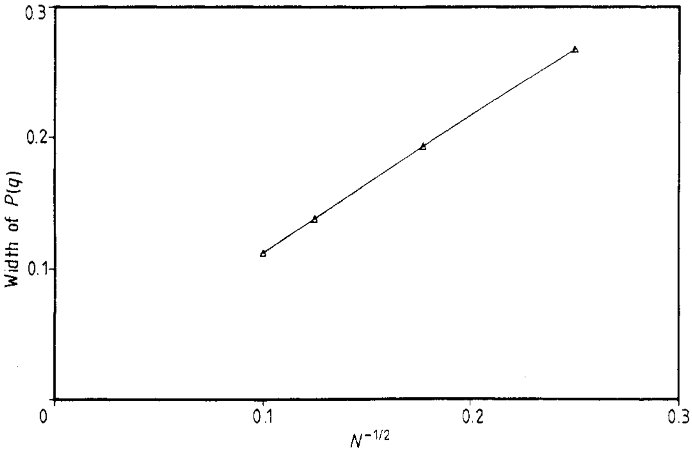

Figure 6. Standard deviation of $P_N(q)$ against $N^{-1/2}$ at $h_{ext} = 1.5$ and $T = 1.4245$ which is above the AT line.

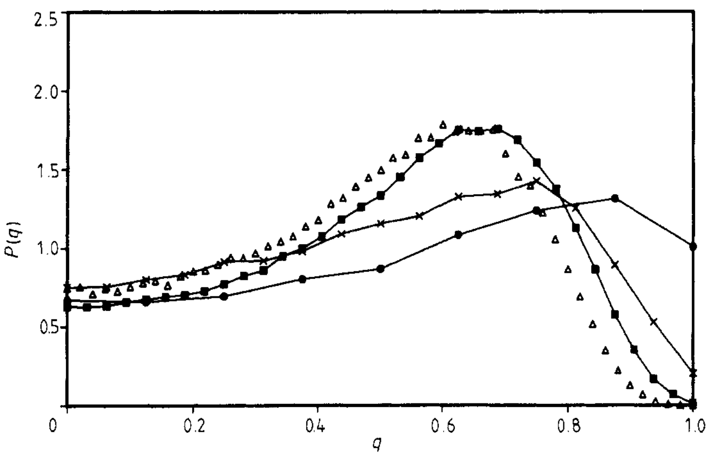

Figure 7. $P(q)$ with $h_{ext} = 0$ at $T = 0.8$ for $N = 16$ (circles), 32 (crosses), 64 (squares) and 100 (triangles).

the results of $P(q)$ for various system sizes N. In addition to the peaks at large q, the distribution functions look much broadened. $P(q)$ has a tail down to $q \sim 0$ and this effect is independent of N. This means that there is finite probability of having states with overlaps close to zero. Furthermore, the shape of the $P_N(q)$ cannot be fitted to a Gaussian function for finite N and the possibility that $P(q)$ is a δ function is ruled out. Since a single pure state implies $P(q)$ is a single δ function, and in this case $P(q)$ is certainly not composed of a finite number of δ functions, thus there should be more than one pure state. Hence the many-pure-state (possibly infinite) phase space structure is strongly suggested from our results below the AT line.

The peak in $P_N(q)$ gets narrower as N increases, especially on the high-q side suggesting that $P(q)$ is zero for q greater than some q_{max} as $N \to \infty$. This suggests that $P(q)$ might be of the form

$$P(q) = C(q) + A(\delta(q - q_{max}) + \delta(q + q_{max})) \tag{3.7}$$

where $C(q) = C(-q)$ is a continuous function for the tail of the distribution and $C(q) = 0$ for $q > |q_{max}|$ and A is some quantity which is independent of q and normalises $P(q)$.

One can extrapolate the value of q_{max} for $N \to \infty$ from figure 7; we get $q_{max} \sim 0.59$. It is interesting to compare with the result of the approximate formula [20]

$$q_{max}(T) = 1 - 2(T/T_c)^2 + (T/T_c)^3 \tag{3.8}$$

which correctly gives the first three terms in the expansion about T_c. With $T_c = 1.52$ in this case (3.8) gives $q_{max} = 0.59$ which agrees very well with the result from simulations.

The form of $P(q)$ suggested in (3.7) is qualitatively the same as for the SK model below the AT line. However, because of the many order parameters at low temperatures in the present model, the scheme of replica symmetry breaking is much more complicated and not known. A relation of the same type as (2.9) in the SK model is not known for this model which renders further quantitative analysis difficult.

3.1.6. At low temperatures. We also performed simulations for temperatures much below T_c. Figure 8 is the result for $P(q)$ for $N = 64$ at various temperatures, all of which show a long tail extending to $q = 0$ with a finite weight. Even though we cannot go to too low a temperature, it seems that the many-state picture persists not only just below T_c, but even down to low temperatures.

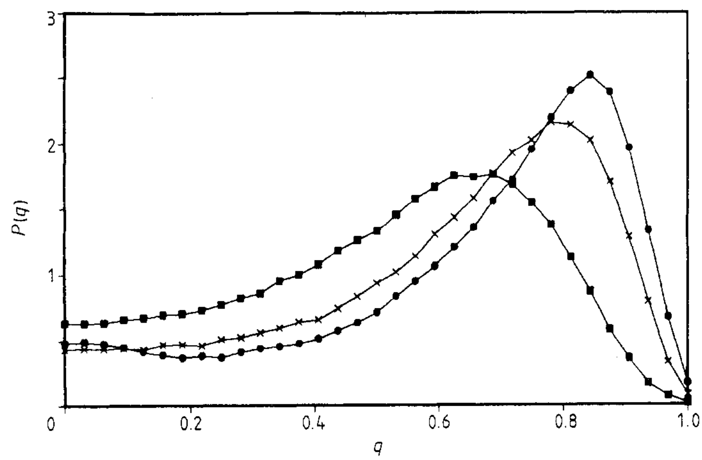

Figure 8. $P(q)$ with $N = 64$ and $h_{ext} = 0$ at $T = 0.8$ (squares), 0.6 (crosses) and 0.5 (circles).

3.2. Bethe lattice

We simulate a spin glass on a Cayley tree with two different boundary conditions and obtain the distribution of the overlap function $P(q)$. Here we also study the case with $c = 3$.

3.2.1. Fixed boundary condition. The boundary sites are fixed arbitrarily in an uncorrelated way and are not allowed to fluctuate thermally. This is equivalent to fixing the boundary conditions in an uncorrelated manner. To try to mimic the Bethe lattice as best as we can, we discard the boundary and one layer in from the boundary in taking the thermal ensemble. The number N quoted in the figure captions is the number of sites in the interior which is only about 25% of the total number of sites of the Cayley tree under consideration. The $P(q)$ at a low temperature below transition for various sizes of the Cayley tree is shown in figure 9. As the size of the lattice increases, $P(q)$ becomes sharper and an approach to a δ function is suggested by the figure. The skewness of the peak is probably due to the fact that too few layers are discarded from the boundary. The single δ function of $P(q)$ supports the picture of a single-state phase space and hence the replica symmetric solution is the correct solution. This agrees with the analytic study in [12].

3.2.2. Closed boundary condition. In this case correlations between different branches are put in by hand by folding up the boundary sites together. Figure 10 illustrates how this is done. Half of the sites on the boundary are gone. All sites are allowed to fluctuate thermally. In taking the thermal ensemble, sites on the boundary and one internal layer are discarded. Figure 11 shows the $P(q)$ for various temperatures. At

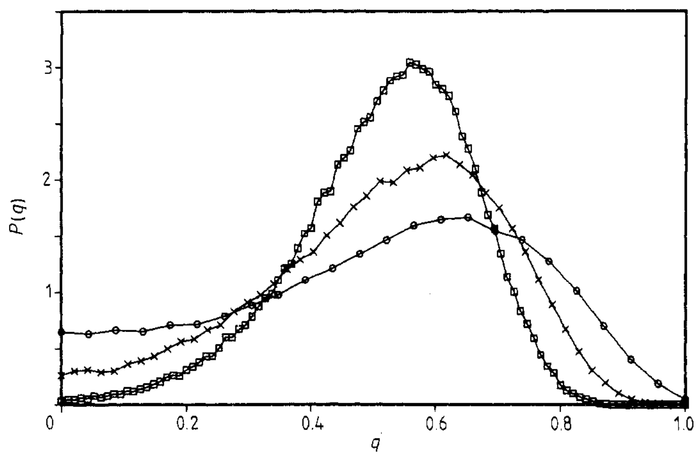

Figure 9. $P(q)$ with fixed boundary condition at $T = 0.8$ for $N = 46$ (circles), 94 (crosses) and 190 (squares).

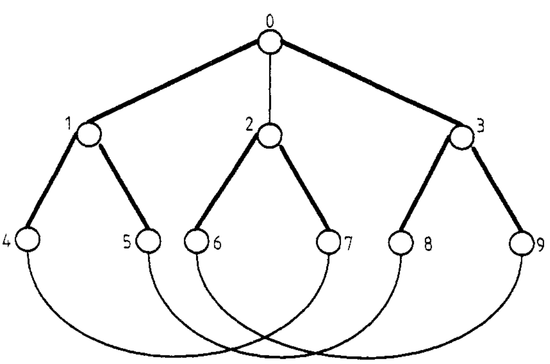

Figure 10. Cayley tree with two levels. To close the tree, site 7 is identified as site 4, 8 as 5 and 9 as 6.

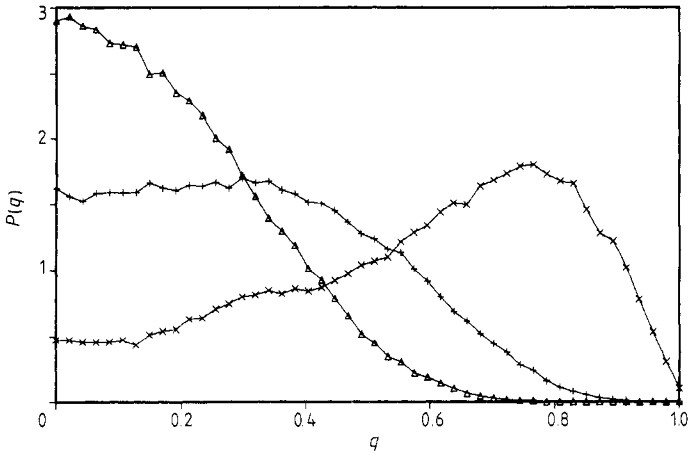

Figure 11. $P(q)$ with closed boundary condition for $N=94$ at $T=1.0$ (triangles), 0.8 (plus signs) and 0.6 (crosses).

high temperature, $P(q)$ peaks at $q=0$, as the temperature is lowered $P(q)$ becomes rather flat and has a slight peak at a large value of q. Figure 12 shows $P(q)$ at low temperature for various sizes. $P(q)$ is non-trivial at low temperature; it has a non-zero tail extended to $q=0$ and this tail persists for all sizes. This reminds one of the feature of the many-state phase space structure as in the spin-glass phase of the SK model. Hence, it seems to suggest the RSB picture when correlations between different branches are introduced. It is clear that the boundary condition has a strong effect on the distribution of the overlap and hence on replica symmetry breaking.

It was shown [13] that RSB occurs at least near and below the transition and our results suggest that with correlated boundary conditions, $P(q)$ seems to indicate RSB. Also, for uncorrelated boundary conditions, $P(q)$ suggests a replica symmetric solution [12]. These results are of course subject to the limitations of our simulation, i.e. finite-size effects and the fact that only two layers have been discarded.

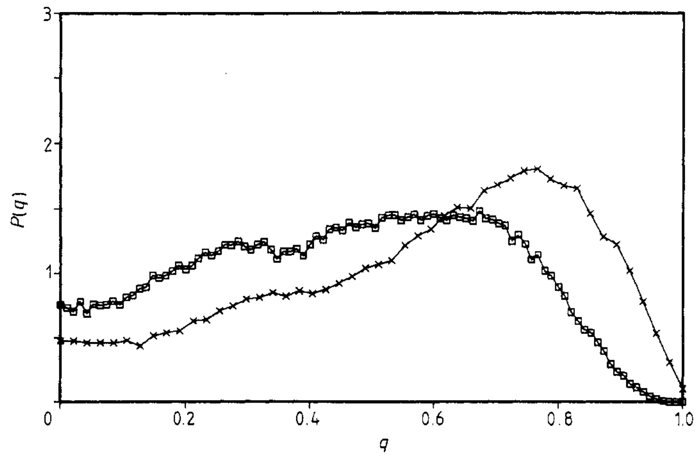

Figure 12. $P(q)$ with closed boundary condition at $T=0.6$ for $N=94$ (crosses) and 190 (squares).

4. Conclusion

We simulated the spin-glass model on a random lattice with a finite average number of neighbours. The spin-glass transition is observed. The analytic results near T_c are confirmed, RSB is quantitatively the same as for the SK model. Above the AT line, $P(q)$ is self-averaging and is a single δ function which supports the single-pure-state picture. Below the AT line, $P(q)$ is non-trivial and non-self-averaging which resembles the many-pure-state picture as in the spin-glass phase of the SK model. This qualitative resemblance seems to hold down to low temperatures. For the Bethe lattice, with fixed boundary condition, $P(q)$ appears to converge onto a δ function even at low temperatures and hence suggests a replica symmetric solution. On the other hand, with the closed boundary condition, correlations between different branches being enforced, a non-trivial $P(q)$ with a non-zero weight extending to $q=0$ is obtained at low temperatures which roughly has the same qualitative picture as the RSB case of the SK model.

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